Diffraction of a weak shock with vortex generation

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SUMMARY

The region of finite vorticity near the edge of a diffracting wedge is investigated. Dimensional analysis gives the dependence of the circulation and the velocity of the vortex region on the pulse strength. A close estimate of the magnitude of these quantities is obtained by replacing the vortex region by a single concentrated vortex. The theoretical conditions at the sharp edge are discussed and compared with observations of real fluid behaviour. A short account of the theory of the core of the spiral vortex sheet in a perfect fluid is appended.

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1. FORMULATION OF THE PROBLEM

Observation of the diffraction of shock waves by sharp wedges shows the existence of a region of finite vorticity near the edge. The phenomenon is shown in plate 1, which reproduces one of several shadowgrams taken at the University of Michigan (Uhlenbeck 1950). The purpose of the present paper is to estimate the position, the velocity and the total circulation of the vortex region, in the limiting case of weak shocks.

On using the acoustic approximation for weak pulses, the classical diffraction problem of Sommerfeld is obtained if the generation of vorticity is ignored. The solution of this problem for non-zero wedge angles was given by Friedlander (1946) and, in a modern and strongly simplified form, by Keller & Blenk (1951) and Miles (1952). This solution exhibits (in general) an infinite velocity near the sharp edge; this fact indicates why the vortex-free solution is not found in reality.

Experience shows that instead of flow past the sharp edges with infinite velocity, a discontinuity sheet forms, originating at the edge, and the velocity at the corner remains finite. Viscosity must be essential for this deviation from the vortex-conservation laws of inviscid flow; however, the problem seems amenable to the now-classical aerodynamic technique in which the detailed mechanism of vortex production is by-passed and the flow with vortex sheet treated by inviscid fluid theory. To do this, the Kutta–Joukowski condition is formulated: vorticity is produced at any instant at such a rate that the resultant flow, with the velocities induced by the vortex elements taken into full account, possesses a finite velocity at the corner. With this condition a problem is obtained which, at least for incompressible flow, is soluble in principle, although the actual calculations may lead to formidable difficulties. (Arguments about the existence of such solutions will be given later.) Qualitatively, however, by considering the motion of newly produced vorticity under the influence of the vorticity generated previously, it is easily understood that the discontinuity sheet will be wound into a narrow spiral, outside of which the flow may be taken as irrotational. Compressibility effects will complicate the phenomenon further, especially towards the core of the vortex; this point will be discussed later.

On the basis of the foregoing arguments, it is proposed to treat the problem by neglecting viscosity (and heat conduction). This assumption has strong support from experimental evidence. In inviscid flow, the diffraction pattern of a sharp pulse (of any strength) at an infinite wedge is exactly similar to itself for all times t. With a coordinate system x, y centred at the vertex, the velocity, the pressure, and other functions of state become functions of x/t and y/t only. This 'quasi-steady' or 'conical' property of the flow field is strikingly well realized, as flow pictures taken at consecutive times show, for diffraction patterns with vortices present. Scrutiny of the experimental data reveals only a small deviation from quasi-steady flow, which is accounted for by viscous effects, and for the purpose of the present theory the inviscid assumption appears well justified.

Next, in view of the restriction to weak pulses, it is proposed to treat the diffraction problem with vortex separation in the linearized approximation, i.e. a potential ϕ will be assumed, satisfying the wave equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial\phi^2}{\partial y^2}=\frac{1}{a^2}\frac{\partial^2\phi}{\partial t^2} \tag{1}$$

(a = velocity of sound, a constant). For quasi-steady solutions, any derivative of the potential ϕ will be a function of $\xi = x/at$ and $\eta = y/at$ only; therefore, ϕ has the form

$$\phi = at\, f(\xi,\eta). \tag{2}$$

Introduction of these variables into (1) leads to the equation

$$f_{\xi\xi}(1-\xi^2)+f_{\eta\eta}(1-\eta^2)-2\xi\eta f_{\xi\eta}=0. \tag{3}$$

Observations show (see plate 1), and the subsequent calculations verify, that, for weak shocks, the vortex region is limited to a domain whose radius is small compared with the radius at of the diffraction region. Here, $|\xi|\ll 1$ and $|\eta|\ll 1$, so that (3) may be well approximated by Laplace's equation,

$$f_{\xi\xi}+f_{\eta\eta}=0. \tag{4}$$

Thus the acoustic approximation leads to incompressible flow in the central region of a quasi-steady flow field.

The incompressible treatment of the central region appears well justified for the problem with vortex generation, as the resultant velocities remain finite due to the superimposed effect of the vortex sheet. Experience shows, however, that appreciable density changes occur towards the core of the vortex spiral. These density changes were investigated experimentally and theoretically by Howard & Matthews (1955). From their results it may be seen that, in the limit of weak shocks, the density variation vanishes sufficiently rapidly to justify an incompressible treatment. Furthermore, for moderate shock strengths, density changes near the vortex do not necessarily invalidate the subsequent estimates based on incompressible flow. In order to be able to calculate the vortex motion under the influence of self-induced velocities in the presence of the wedge by means of the approximation of incompressibility, we need assume only that the Mach number of the flow velocities along the wedge surface remain small compared with unity.

2. Solution by dimensional analysis

The linearized potential ϕ for the diffraction problem with vortex separation, as a solution of the wave equation (1), can be considered as a sum of two solutions:

$$\phi = \phi_S + \phi_\Gamma, \tag{5}$$

where ϕ_S is the vortex-free (Sommerfeld) solution, and ϕ_Γ is an additional part due to the existence of the vortex sheet. The potential ϕ_S is known; it depends on the strength of the pulse, the incidence of the oncoming wave, and the wedge angle. The dependence on the strength of the pulse can be given immediately. Let u_0 be the gas velocity in the rear of the undisturbed incident pulse. (This 'afterflow velocity' is taken positive in the direction of the incident wave propagation.) All velocity components derived from the solution will be proportional to u_0, so that one can write, with a simple renormalization of (2),

$$\phi_S = a u_0\, t f_S(\xi, \eta), \tag{2a}$$

where f_S is dimensionless and depends on the flow geometry, i.e. on the wedge angle and the angle of incidence only.

The additional solution ϕ_Γ naturally also depends on u_0 and the flow geometry; however, its dependence on u_0 will not be as simple as that expressed by (2a). It is clear that the position of the vortex relative to the 'Mach circle' (with radius at) will depend on the pulse intensity u_0, and thus the velocities derived from ϕ_Γ will not be simply proportional to u_0. Using the fact that for $u_0 \ll a$ the vortex region is very near to the vertex of the wedge, the dependence of ϕ_Γ on u_0 will be found by dimensional analysis.

The function f_S in (2a) is a solution of equation (3), and is known. For $|\xi| \ll 1$, $|\eta| \ll 1$, f_S must become a solution of Laplace's equation (4). Therefore,

$$f_S = \mathscr{R}\{F_S(\xi + i\eta)\} \equiv \mathscr{R}\{F_S(\zeta)\}, \tag{6}$$

for $|\zeta| \ll 1$. It is easy to write down the form of the function $F_S(\zeta)$, as it must represent the complex potential for incompressible flow past an infinite wedge. Let the wedge with the angle β be formed by the positive real axis and ray with the argument $-\beta$. The function F_S takes the form

$$F_S(\zeta) = A_1 \zeta^n + A_2 \zeta^{2n} + \dots, \tag{7}$$

where

$$n = \frac{\pi}{2\pi - \beta}, \tag{8}$$

so that, for $0 \leqslant \beta < \pi$, n lies in the range $\frac{1}{2} \leqslant n < 1$. The dimensionless quantities $A_1, A_2 \dots$ become real for the wedge position specified above; they depend on the flow geometry and can be given only if the Sommerfeld problem is solved, i.e. if f_S is found as a solution of the full equation (3). Expansion of the Sommerfeld solution around $\zeta = 0$ naturally confirms the form of (7), but only the first two terms represent analytic functions of the present variable ζ. This means that if more terms were needed for the treatment of the vortex generation problem, the use of the approximate equation (4) would be unjustified.

As $\frac{1}{2} \leqslant n < 1$, only the first term in (7) represents a 'singular' flow past the wedge with infinite velocities, which is taken as the 'cause' of vortex separation. It may be imagined that the whole phenomenon of vortex generation is not only 'caused', but also fully 'dominated' by the first term of (7), which would then be the only one that need be kept for the subsequent analysis. This assumption will be made now, and its justification will be sought *a posteriori*.

Returning to dimensional variables, put

$$\phi_S = a u_0 t F_S, \qquad z = a t \zeta, \tag{9}$$

so that

$$\begin{aligned} \phi_S &= a u_0 t \left\{ A_1 \left(\frac{z}{at}\right)^n + A_2 \left(\frac{z}{at}\right)^{2n} + \dots \right\} \\ &= a^{1-n} u_0 A_1 z^n t^{1-n} + a^{1-2n} u_0 A_2 z^{2n} t^{1-2n} + \dots \\ &\equiv K_1 z^n t^{1-n} + K_2 z^{2n} t^{1-2n} + \dots \end{aligned} \tag{10}$$

If only the first term plays a role, then there is only one physical constant in the problem: K_1, which has the dimensions of a velocity to the power $2 - n$.

Now consider the vortex region, which is 'generated and dominated' by the first term of the flow potential (10), and let it be characterized by the total circulation Γ, the position of the core (centre of the spiral) z_Γ, and the velocity w_Γ of the point z_Γ. As the resultant flow is also quasi-steady, the time-dependence of these quantities is given as follows:

$$z_\Gamma \propto t, \qquad w_\Gamma \propto t^0, \qquad \Gamma \propto t. \tag{11}$$

The dimensional factors multiplying these time-dependencies can be formed only by the quantity K_1, and in only one way:

$$z_\Gamma \propto K_1^{1/(2-n)} t, \qquad w_\Gamma \propto K_1^{1/(2-n)}, \qquad \Gamma \propto K_1^{2/(2-n)} t, \tag{12}$$

or, using the expression for K_1 in terms of a, u_0 and A_1,

$$\left.\begin{aligned} z_\Gamma &\propto A_1^{1/(2-n)}\left(\frac{u_0}{a}\right)^{1/(2-n)} at, \\ w_\Gamma &\propto A_1^{1/(2-n)}\left(\frac{u_0}{a}\right)^{1/(2-n)} a, \\ \Gamma &\propto A_1^{2/(2-n)}\left(\frac{u_0}{a}\right)^{2/(2-n)} a^2 t = A_1^{2/(2-n)}\left(\frac{u_0}{a}\right)^{n/(2-n)} au_0\, t. \end{aligned}\right\} \tag{13}$$

The use of the factor $A_1^{1/(2-n)}$ does not make much sense at present, as a further factor dependent on the flow geometry is implied in (13). But (13) gives the correct dependence on the 'afterflow Mach number' u_0/a ($\equiv M_0$), which characterizes the pulse strength. Indeed, if a and u_0 are not independent constants of the problem but occur only in the combination K_1, then (13) represents the only possible solution.

The explicit dependence of ϕ_Γ on M_0 is given as $\phi_\Gamma \propto \Gamma$, so that

$$\phi_\Gamma \propto M_0^{n/(2-n)} au_0\, t f_\Gamma\left(\frac{\zeta}{M^{1/(2-n)}}\right)$$

for $|\zeta| \ll 1$.

The validity of the original assumption that the first term of the expansion (10) is predominant, can now be shown in an 'iterative' way. Introduce z_Γ from (13) into (10) and compare the magnitudes of the different terms. The Nth term becomes

$$\begin{aligned} \phi_{S,N}(z_\Gamma) &= au_0 t A_N \left(\frac{z_\Gamma}{at}\right)^{Nn} \\ &\propto au_0\, tA_N\, A_1^{N/(2-n)} M_0^{Nn/(2-n)}. \end{aligned} \tag{14}$$

It is seen that the magnitude of the terms diminishes rapidly with increasing N if M_0 is small. Thus, in the limiting case of weak pulses, the vortex region will really be 'generated and dominated' by the first term. Howard & Matthews (1955) found that the corresponding dependence $z_\Gamma/at \propto M_0^{1/(2-n)}$ was excellently verified by their own experiments.

It must be emphasized that the simple result obtained for diffraction is not always valid for vortex generation problems; indeed, it is quite exceptional. In wing theory there is considerable interest in problems of vortex separation from sharp edges. The method generally used is the replacement of the vortex region by a single concentrated vortex, as will be discussed shortly. Only one case in wing theory corresponds to the diffraction problem: for a rectangular thin wing in supersonic flow, the vortex separation from the side edge is perfectly analogous to the diffraction by a wedge with $\beta=0$. This case was treated by Cheng (1955), using the single-vortex model, and his results agree with (13). Other wing problems can be related to the diffraction case only if the wedge is supposed to move, with some constant velocity v_0. This causes additional complications, since the phenomenon depends strongly on the direction of the motion of the wedge.

The simplest cases are obtained for wedge motion in the direction of the bisectrix of the wedge angle, as otherwise the wedge motion causes vortex separation even without the diffracted pulse. As an example, the case of zero wedge angle ($\beta = 0$, $n = \frac{1}{2}$) will be considered. The potential of the constant flow relative to the edge is simply $v_0 z$. If $v_0 \ll a$, a development of ϕ_S relative to the moving edge can be expected to have the same form, (10), as for $v_0 = 0$, except that the A_1, A_2, . . become functions of v_0/u_0, and there will be an additional term of the form

$$\Delta\phi_S = v_0\, z$$

due to the proper motion of the plate. Now, the 'iterative' justification of the predominance of the first term in (10) fails, because of this additional term. From (13), with $n = \frac{1}{2}$, it follows that

$$\Delta\phi_S(z_\Gamma) \propto a v_0\, A_1^{2/3}\left(\frac{u_0}{a}\right)^{2/3}.$$

Equation (14) yields, for $N = 1$,

$$\phi_{S,1} \propto a u_0 t A_1^{5/3}\left(\frac{u_0}{a}\right)^{1/3},$$

which is larger than the previous term only if

$$\frac{1}{A_1}\frac{v_0}{u_0}\left(\frac{u_0}{a}\right)^{1/3} \ll 1.$$

If v_0 is kept constant, this condition clearly fails in the limit $u_0 \to 0$. Physically, this means that for a weak shock, and $v_0 \gg u_0$, the vortex is swept away from the edge by the velocity v_0. If, however, v_0 and u_0 have the same order of magnitude, and u_0/a_0 is still small, (13) becomes valid again.

It may be noted that, for a wedge flow with 'generalized' quasi-steadiness such that

$$\phi_{S,1} = K_1\, t^{2m-1}\left(\frac{z}{t^m}\right)^n \tag{15}$$

near $z = 0$, a result corresponding to (12) may be obtained, if K_1 is the only important physical constant:

$$z_\Gamma \propto K_1^{1/(2-n)} t^m, \qquad w_\Gamma \propto K_1^{1/(2-n)} t^{m-1}, \qquad \Gamma \propto K_1^{2/(2-n)} t^{2m-1}. \tag{16}$$

3. The 'single vortex' approximation

In order to find the numerical factors still missing in (13), use will now be made of an approximation mentioned previously: the whole vorticity region will be replaced by a single concentrated vortex. The results already obtained by dimensional analysis will incidentally be re-established.

The complex potential near the vertex will be decomposed again into a vortex-free part and an additional solution due to the vortex:

$$\phi = \phi_S + \phi_P. \tag{5a}$$

According to the results of § 2, put

$$\phi_S = a u_0\, t A_1 \zeta^n. \tag{10a}$$

For the single-vortex model, ϕ_Γ is the potential for a vortex of strength Γ situated at the point $z_\Gamma = at\zeta_\Gamma$, in the presence of the wedge, which must be a streamline of the flow ($\mathscr{I}\{\phi\} =$ constant). The solution is

$$\phi_\Gamma = \frac{i\Gamma}{2\pi} \log \frac{\zeta^n - \zeta_\Gamma^n}{\zeta^n - \bar{\zeta}_\Gamma^n}. \tag{17}$$

For this potential, the velocity vanishes as $|\zeta| \to \infty$. It is easy to realize that this is the proper boundary condition at infinity. The quantities ϕ_S and ϕ_Γ are, respectively, the approximations for $|\zeta| \ll 1$ to the solutions ϕ_S and ϕ_Γ of the full wave equation. The Sommerfeld solution ϕ_S takes care of all the boundary conditions along the Mach circle, imposed by conditions of proper joining to the pulses outside the Mach circle, these joining conditions are naturally the same for the solution ϕ (with vorticity) and for ϕ_S, so that the solution $\phi_\Gamma = \phi - \phi_S$ fulfills 'zero disturbance' conditions along the Mach circle. Thus, the solution ϕ_Γ expresses the sole effect of the growing vortex region, and the vanishing of the solution on the Mach circle simply expresses the fact that this disturbance, originating at $z = 0$, $t = 0$, can spread only with the velocity of sound. Evidently, as long as the incompressible approximation can be made, i.e. as long as $z_\Gamma \ll at$, it makes no difference whether the 'no-flow' condition is fulfilled on the Mach circle or at infinity.

From the total potential

$$\phi = au_0\, tA_1\left(\frac{z}{at}\right)^n + \frac{i\Gamma}{2\pi} \log \frac{z^n - z_\Gamma^n}{z^n - \bar{z}_\Gamma^n} \tag{18}$$

the complex velocity w_K 'at' the singular point z_Γ will be derived. This is defined, according to Kirchhoff, as the regular part of the complex velocity at z_Γ,

$$w_K = \lim \left(\frac{\partial \phi}{\partial z} - \frac{i\Gamma}{2\pi(z - z_\Gamma)}\right), \text{ as } z \to z_\Gamma. \tag{19}$$

The calculation of the limit, which requires some care, leads to the following result:

$$w_K = \frac{n}{\zeta_\Gamma}\left\{u_0\, A_1\, \zeta_\Gamma^n - \frac{i\Gamma}{2\pi at}\left(\frac{1-n}{2n} + \frac{\zeta_\Gamma^n}{\zeta_\Gamma^n - \bar{\zeta}_\Gamma^n}\right)\right\}. \tag{20}$$

If the vortex is free, it moves with the Kirchhoff velocity w_K, that is

$$w_K = \frac{d\bar{z}_\Gamma}{dt} = a\bar{\zeta}_\Gamma. \tag{21}$$

(Note that the complex velocity is the conjugate of the physical velocity vector.) Equations (20) and (21) together give an important connection between Γ and ζ_Γ.

A further complication arises from the fact that this *single* vortex with *variable* intensity is not free; strictly speaking, it is incompatible with the vortex conservation laws. The original vortex spiral naturally is free, i.e. it does not sustain any forces. By concentration of the spiral into a single vortex, its 'umbilical chord' has been cut, so that either the vortex strength cannot grow, or, if Γ changes, an unbalanced pressure jump will be found on any line connecting the vertex of the wedge and the vortex. This pressure jump is expressed by the fact that $\partial\phi/\partial t$ has a multivalued (logarithmic) term for variable Γ; the magnitude of the jump is $\rho d\Gamma/dt =$ constant. Due to the 'hydrostatic' character of this jump pressure, the resultant force is the same on any line connecting the vertex and the vortex, and is equal to $|z_\Gamma|\rho\Gamma/t$; its direction is perpendicular to z_Γ.

It would seem that the single-vortex model has to be abandoned, due to its inconsistency with vortex-conservation laws. However, an ingenious proposal which enables this useful approximation to be retained has been made by Brown & Michael (1954) and by Edwards (1954): let the single vortex not be free, but subject to a Joukowski force, which cancels the unbalanced force on the vertex-vortex line. (Only an unbalanced moment will remain.) In order to present this condition in full generality, let the sheet be generated at a point z_0, so that the (complex) force to be balanced is

$$F_1 = i\rho(z_\Gamma - z_0)\frac{d\Gamma}{dt}. \tag{22}$$

The Joukowski force is

$$F_2 = -i\rho\left(\bar{w}_K - \frac{dz_\Gamma}{dt}\right)\Gamma. \tag{23}$$

so that the condition of the vanishing total force leads to

$$\bar{w}_K = \frac{1}{\Gamma}\left\{\frac{d(z_\Gamma\Gamma)}{dt} - z_0\frac{d\Gamma}{dt}\right\}. \tag{24}$$

It has been pointed out by Cheng (1955) that this equation may be used for any flows with vortex generation, even without similarity. In the general similar case expressed by (16), $z_\Gamma \propto t^m$ and $\Gamma \propto t^{2m-1}$, so that (24) yields (with $z_0 = 0$)

$$w_K = \frac{3m-1}{t}\bar{z}_\Gamma, \tag{25}$$

or, for quasi-steady flow ($m=1$),

$$w_K = 2a\bar{\zeta}_\Gamma, \tag{26}$$

which replaces (21). It is seen that (21) and (26) differ only by a numerical factor 2; the exact number to be taken is naturally unknown as long as the true spiral structure is undetermined, but (26) may be supposed to lead to a close estimate. The 'free-vortex' condition

$$w_K = \frac{d\bar{z}_\Gamma}{dt} = \frac{m}{t}\bar{z}_\Gamma. \tag{27}$$

leads to the same result as (25) for $m=\frac{1}{2}$, when the single-vortex model becomes exact: $\Gamma \propto t^{2m-1} =$ constant. This interesting limiting case, which may be realized in special cases of incompressible flow, is certainly worthy of further investigation.

Returning to our problem, the elimination of w_K from (20) and (26) leads to the following connection between ζ_Γ and Γ:

$$\frac{2}{n}\zeta_\Gamma\,\bar{\zeta}_\Gamma = M_0\,A_1\,\zeta_\Gamma^n - \frac{i\Gamma}{4\pi a^2 t}\left\{\frac{1}{n} + \frac{\zeta_\Gamma^n + \bar{\zeta}_\Gamma^n}{\zeta_\Gamma^n - \bar{\zeta}_\Gamma^n}\right\}, \tag{28}$$

Putting $\zeta_\Gamma = \sigma \exp(i\vartheta)$, the imaginary and real parts of (28) yield, respectively,

$$0 = M_0\,A_1\,n\sigma^n \sin n\vartheta - \frac{\Gamma}{4\pi n a^2 t}, \tag{29}$$

and

$$\frac{2}{n}\sigma^2 = M_0\,A_1\,\sigma^n \cos n\vartheta - \frac{\Gamma}{4\pi a^2 t}\cot n\vartheta. \tag{30}$$

Elimination of Γ in (30) by (29) leads to the equation

$$\frac{2}{n}\sigma^2 = M_0\,A_1\,\sigma^n(1-n)\cos n\vartheta,$$

or finally to

$$\sigma \equiv \frac{|z_\Gamma|}{at} = \left(\frac{n(1-n)}{2}\cos n\vartheta\,A_1\,M_0\right)^{1/(2-n)} \tag{31}$$

and

$$\Gamma = 8\pi a^2 t\,\frac{\tan n\vartheta}{1-n}\,\sigma^2. \tag{32}$$

Equation (31) still contains the factor A_1, which has to be taken from the development of the known Sommerfeld solution. (This is the only point where this function will be explicitly needed.) From the solution of Keller & Blenk (1951) and Miles (1952) it is found that

$$A_1 = \frac{2^{2-n}}{\pi(1-n)}\sin n\pi \sin n\alpha,$$

where α is the angle between the wedge bisectrix and the incident wave normal. With this value, (31) becomes

$$\sigma = 2\left\{\frac{n \sin n\pi}{2\pi}\sin n\alpha \cos n\vartheta\,M_0\right\}^{1/(2-n)}. \tag{33}$$

Equation (33) expresses the relative distances of the vortex from the vertex as a function of the wedge angle (n), the incident wave direction (α), the incident shock strength (M_0), and the direction of the vortex motion ϑ, measured from the (real axis) side of the wedge. The angle ϑ still being unknown, (33) provides only a semi-empirical formula for σ and, by (32), for Γ. Equation (33) was tested by Howard & Matthews (1955), who found the calculated values consistently about 10% too low, for $\beta = 5°$. Similarly good results were obtained by Waldron (1954).

It may be pointed out that even at this stage, with ϑ still undetermined, it is easy to find upper bounds for quantities of interest. For instance, according to (32) and (33),

$$\Gamma \propto \sin n\vartheta(\cos n\vartheta)^{n/(2-n)},$$

which has a maximum at

$$\sin^2 n\vartheta = 1 - \frac{n}{2}.$$

An upper bound for the circulation can be found with this value of ϑ.

4. The Kutta condition

The condition that the velocity remains finite at the sharp edge can be applied also to the single-vortex model; it determines the direction ϑ of the vortex motion, as there is only one ϑ for which the velocity at the origin remains finite. It will be seen, however, that the most serious shortcomings of the single-vortex approximation present themselves in connection with the Kutta condition. The previous calculations herein have been pushed as far as possible without making use of this requirement.

Near the origin, the potential (18) can be developed in the following power series:

$$\phi = au_0 tA_1 \zeta^n + \frac{i\Gamma}{2\pi}\sum_{N=1}^{\infty}\frac{(-1)^N}{N}(\zeta_\Gamma^{-nN} - \bar{\zeta}_\Gamma^{-nN})\zeta^{nN}. \tag{34}$$

The velocity remains finite for $\zeta = 0$ if the coefficient of ζ^n vanishes, or, putting $\zeta_\Gamma = \sigma \exp(i\vartheta)$ again, if

$$M_0 A_1 = \frac{\Gamma}{\pi a^2 t \sigma^n} \sin n\vartheta. \tag{35}$$

Equations (29) and (35) immediately yield

$$4n \sin^2 n\vartheta = 1. \tag{36}$$

According to this equation, ϑ depends on the wedge angle only. Typical values are:

β	n	ϑ
0	1/2	90°
90°	2/3	56·6°
180°	1	30°

Howard & Matthews (1955) observed for $\beta = 5°$ the value $\vartheta = 68°$, while the theoretical angle according to (36) is $\vartheta = 88°$. For larger wedge angles the disagreement becomes even worse. The experiments made at the University of Michigan with 90°-wedges (plate 1) show values of ϑ of about 32° (depending slightly on the angle of incidence) instead of 56·6°. A critical examination of the Kutta condition for the single-vortex model is warranted; it will lead to an explanation of the growing difficulties for increasing wedge angle.

For this purpose, it is even necessary to investigate the original problem of the vortex spiral. Let $z_\lambda = at\zeta_\lambda$ be the location of the vortex sheet, given as a function of a suitable parameter, such as the arc length λ of the spiral (measured from the origin). Let Γ_λ be the circulation of the spiral beyond the point z_λ; the total circulation in the whole spiral, $\Gamma_{\lambda=0}$ will be designated from now on by Γ_0. The line density of vorticity is, with the present sign conventions, $-d\Gamma_\lambda/d\lambda$. By superposition, the potential of the whole spiral flow may be expressed with the help of the potential (18) as 'elementary solution':

$$\phi = au_0 tA_1 \zeta^n - \frac{i}{2\pi}\int_\lambda \log\frac{\zeta^n - \zeta_\lambda^n}{\zeta^n - \bar{\zeta}_\lambda^n}\, d\Gamma_\lambda. \tag{37}$$

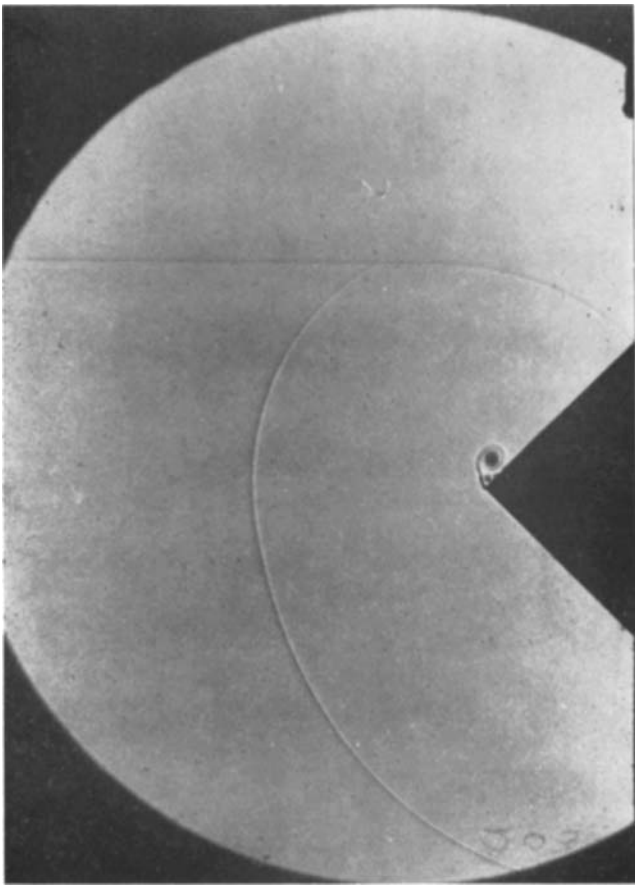

Plate 1. Shadowgram of a shock wave diffracted by a 90° wedge. $M_0=0{\cdot}27$, $t=152$ microseconds. Ahead of the shock, the air is under atmospheric conditions. (Reproduced, by kind permission of Prof. O. Laporte, from a report prepared at the University of Michigan.)

The Kutta condition now can be expressed as follows:

$$au_0 tA_1 = -\frac{i}{2\pi}\int_\lambda (\zeta_\lambda^{-n} - \bar{\zeta}_\lambda^{-n})\, d\Gamma_\lambda = -\frac{1}{\pi}\int_\lambda \frac{\sin n\vartheta_\lambda}{\sigma_\lambda^n}\, d\Gamma_\lambda. \tag{38}$$

It is seen that (35) approximates (38) in a rough way, as parts near the origin ($\sigma_\lambda \to 0$) can give a significant contribution to the integral in (38).

Along the spiral, the tangential velocity is discontinuous while the normal velocity is continuous, that is, the real part of the potential (37) will have a jump while the imaginary part (the stream function) is the same on both sides. This behaviour is assured by setting up the solution in the form (37) (with real Γ_λ). If the complex potentials on the two sides of the spiral are called ϕ_1 and ϕ_2, their difference at any point of the spiral is real and given by

$$\phi_2 - \phi_1 = \Gamma_\lambda. \tag{39}$$

Finally, the kinematic and dynamic conditions along the vortex sheet must be formulated. The condition is especially simple at the origin of the sheet, where the velocity normal to the spiral vanishes, and the complex velocities on the two sides of the sheet, w_1 and w_2, are parallel. In view of (39), equality of the pressure on both sides of the sheet can be expressed by the equation

$$\frac{d\Gamma_0}{dt} = \frac{\bar{w}_1 + \bar{w}_2}{2}(w_1 - w_2). \tag{40}$$

If the total circulation changes with time, it is obviously impossible to have both $w_1 = 0$ and $w_2 = 0$.

Let the angles between the sheet and the adjacent side of the wedge at the origin be γ_1 and γ_2 on the two sides of the spiral; $\gamma_1 + \gamma_2 = 2\pi - \beta$. The flow velocity will be zero, finite or infinite at the origin according as the angle γ is less than, equal to, or larger than π, respectively. (It can be shown that the sheet will have infinite curvature at $\zeta_\lambda = 0$; nevertheless, the simple statement above remains true.) It is now evident that one of the corners must form the angle π, and the other the angle $\pi - \beta$. For, if both γ_1 and γ_2 were less than π, both w_1 and w_2 would vanish, which is impossible for $d\Gamma_0/dt \neq 0$. If one corner had an angle larger than π, the velocity would be infinite, and the Kutta condition thereby violated.

Now, two cases have to be distinguished. If $\beta > 0$, the velocity is finite on one side, and zero on the other side, so that, in view of (40),

$$w_2 = 0, \qquad |w_1| = \sqrt{2d\Gamma_0/dt} \quad \text{(say)}. \tag{41}$$

On the other hand, if $\beta = 0$ and $\gamma_1 = \gamma_2 = \pi$, it is impossible to have $w_2 = 0$ and still have a simple spiral flow. Vanishing velocity together with $\gamma_2 = \pi$ is only possible if the potential behaves on one side as z^n with $n = 2, 3, \ldots$ These stagnation points imply a profound change in flow configurations, making the simple spiral impossible. Therefore, (41) must hold for $\beta > 0$ and cannot be true for $\beta = 0$. This is a first indication of an essential difference between the spirals for zero wedge angle and for finite values of β.

This point can be elaborated further as follows. The flow potential of the spiral, as given in (37), can be written, with the help of the Kutta condition (38), in the form

$$\phi = -\frac{i}{2\pi}\int_\lambda \left\{(\zeta^{\ n} - \bar{\zeta}_\lambda^{-n})\zeta^n + \log\frac{\zeta^n - \zeta_\lambda^n}{\zeta^n - \zeta_\lambda^n}\right\} d\Gamma_\lambda. \tag{42}$$

In this expression the integrand represents the complex potential of a flow element in which the vortex flow is 'paired' with a part of the ζ^n-flow in such a way that the Kutta condition is always fulfilled. The velocity at the origin will remain finite even if only a part of the integral in (42) is considered.

Let the domain of integration in (42) be subdivided in two parts: a short length Δ of the spiral, beginning at the origin, and the remaining domain $\lambda - \Delta$. Differentiation of the latter integral will give the velocity of the sheet near $\zeta = 0$ with no contribution from the local sheet element; this is regular and behaves as ζ^{2n-1}. For $\beta = 0$, $n = \frac{1}{2}$, this velocity is (in general) not zero; in particular, it does not vanish for the spiral, as only elements with real ζ_λ have vanishing contributions, and the sign of the velocity depends on the sign of the imaginary part of ζ_λ, which is always the same for the spiral; also, $d\Gamma_\lambda$ never changes its sign. On the other hand, if $\beta > 0$, the contribution from the integral over $\lambda - \Delta$ to the velocity near the origin always vanishes for $\zeta = 0$, and the 'strength' of the zero increases with β.

Now consider the part contributed by the integral over Δ, which will introduce the velocity jump across the sheet at $\zeta_\lambda = 0$. If $\beta = 0$, this jump is superimposed upon an already existing mean velocity. (It can be shown that for $n = \frac{1}{2}$ the contribution from $\lambda - \Delta$ tends towards the mean velocity and the part from Δ gives the velocity difference in (40), for the limit $\Delta \to 0$.) In contrast, for $n > \frac{1}{2}$, the contribution from the small length Δ not only has to produce the velocity difference but the mean velocity as well; it has to change the character of the flow profoundly—from a velocity of the type ζ^{2n-1} to a velocity behaving as ζ^0 on one side and $\zeta^{(2n-1)/(1-n)}$ on the other side of the sheet. It can be seriously doubted whether such a type of solution exists, i.e., whether a spiral solution of the form (42) can be found for $n > \frac{1}{2}$. No such difficulties arise for $\beta = 0$, and it can be shown that the required type of flow around the origin is possible for a sheet which is of the form $\eta_\lambda = \text{const.}\,\xi_\lambda^{3/2}$, for $\zeta_\lambda (\equiv \xi_\lambda + i\eta_\lambda) \to 0$. (The reason is that a velocity field of the type $c_0 + c_1\zeta^{1/2} + \ldots$ can be derived from the integral over $\lambda - \Delta$ and the spiral has to be a streamline of this field for $\zeta_\lambda \to 0$.)

The somewhat academic question of the existence will not be discussed any further, and the results of the previous considerations can be summed up as follows. If $\beta = 0$, the spiral induces at the origin a certain mean velocity which provides a speedy transport of newly created vorticity, stretching the sheet near the origin and reducing the density of the sheet such that its influence near the origin is weak. If $\beta < 0$, on the other hand, the vortex elements nearest to the origin have to induce their own mean 'transport' velocity, which is physically possible only by a strong accumulation of vorticity near the edge, increasing in strength with the wedge angle.

Returning to the original question, it becomes clear why the fulfillment of the Kutta condition becomes a poor approximation for blunt wedges. Mathematically, the question is, how well does (35) approximate (38)? The approximation is poor if a high vortex density near the origin exists and this must be the case for blunt wedges.

The doubts about the existence of the idealized solution (42) for $n > \frac{1}{2}$ suggest a re-examination of the phenomenon in real fluids. This will be based on the shadowgrams taken at the University of Michigan, where experiments were carried out with a pressure ratio of 1·44 of the incident shock. These pulses may be considered weak enough for the point of view adopted in this paper, namely, that the vortex generation problem is essentially an incompressible flow phenomenon. In contrast, experiments by Waldron (1954) with pressure ratios of 1·93 and above show supersonic flow near the origin combined with much more complicated vortex-generation patterns, which will not be discussed here.

The shadowgram shown in plate 1 reveals a startling discrepancy between the classical theoretical assumption—that vorticity is generated at the sharp edge only—and the real fluid-behaviour. The shadowgram shows a *second* vortex, smaller, but very near the edge, rotating in a direction opposite to the sense of the main vortex. A qualitative explanation of this 'secondary vortex separation' is easily found. The flow pattern of the primary vortex spiral, originating at the sharp edge, must have found along the side of the wedge near the vortex a deceleration, or diminishing velocity towards the vertex of the wedge. This is connected with a pressure rise, and therefore the boundary layer along the wall separates and forms the secondary vortex. Thus, secondary vortex separation is a 'viscous' effect, and the basic assumption that the flow can be treated as inviscid also needs re-examination.

A series of shadowgrams of the same diffraction case as that in plate 1, taken at different times, show however that the flow pattern with secondary vortex separation is still essentially quasi-steady. Boundary-layer separation patterns are influenced by the pressure distribution and by the Reynolds number. It is known that domains of Reynolds numbers exist in which the separation phenomenon is determined by the pressure distribution only. In such a domain, explicit use of the viscosity for the vortex *generation* problem is not needed, just as in the case of a sharp edge. (The dissipative effect of vorticity has been already neglected for the primary vortex separation.) Thus the existence of quasi-steady flow patterns with secondary vortex separation is understandable, but it must be borne in mind that these patterns might be different in a different domain of Reynolds numbers. The pertinent Reynolds number can be taken as Γ_0/ν (ν = kinematic viscosity); using for Γ_0 the estimate expressed by (32) and (31), this number becomes (omitting factors depending on flow geometry only)

$$Re = \nu^{-1} a^2 M_0^{2/(2-n)} t, \tag{43}$$

so that, given sufficient time (and size), all values of Re may be attained, with the time unit fixed by the gas properties and the shock strength.

The first two factors represent a number of the order of the molecular collision frequency in the gas, i.e. a number of the order of 10^{10} sec^{-1} for atmospheric air, so that for the case represented by plate 1, $Re \doteqdot 10^5$.

What are the consequences of the secondary vortex separation effect on the previous results? If the flow pattern is very nearly quasi-steady, as is proved by experience at least for a limited *Re*-domain, the dimensional analysis of § 2 remains valid for weak shocks. The numerical factors estimated in § 3 become less accurate, but no change in the orders of magnitude can be expected. The application of the Kutta condition to the single-vortex model becomes obviously senseless, as the secondary vortex is very near to the edge and has an appreciable effect there.

Secondary vortices become increasingly important with growing wedge angles. They are hardly detectable in the pictures taken by Howard & Matthews for $\beta = 5°$. A fine picture of the phenomenon for sharp wedges can be found in Prandtl's article in *Handbuch der Experimental-Physik* (p. 20), also reproduced in Goldstein's *Modern Developments in Fluid Dynamics* (p. 40). The small secondary vortex would be easily overlooked by an observer who is not determined to find it.

Secondary vortices will also be strongly influenced by the motion of the wedge, as discussed in § 2. Three cases are sketched in figure 1 for the case $\beta = 0$; the arrow indicates the velocity v_0 relative to the edge. The strongest secondary vortices will be found in case *c*. In this case, v_0 also reduces the 'mean' velocity at the edge which transports newly created vorticity away from the edge. It may be conjectured that if an inviscid spiral solution for $\beta = 0$ exists mathematically, it can exist only beyond a certain limiting value of v_0.

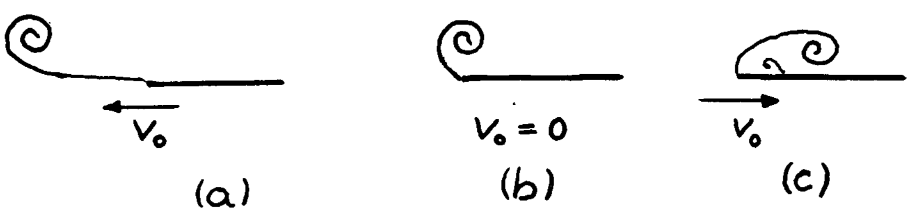

Figure 1. Vortex separation with fluid motion parallel to a sharp edge.

The case represented in figure 1*c* is the one of particular interest in the theory of delta wings. It is seen that the application of the Kutta condition makes no sense for a single-vortex model, from which at most a semi-empirical result can be expected.

Experiments by Michael (1955) on leading-edge separation from delta wings show the existence of secondary vortex separation, and even more complicated phenomena. For small aspect ratios, the 'primary' vortex appears to be broken up into two regions, one very near the tip, and a main vortex, so that three distinct vortices can be found. If the aspect ratio is increased (which corresponds to an increase of v_0 in the diffraction case), the vortices are broken up further, and numerous 'vortical regions' appear. In these cases, agreement with the results calculated from the single-vortex approximation is poor.

Finally, attention should be drawn to the uneven vortex density at the beginning of the sheet in plate 1; other pictures also show 'wiggles' and 'knots' along the spiral. A detailed investigation of the vortex generation in a viscous fluid might show that it is a truly unsteady (not quasi-steady) phenomenon—possibly periodic with a frequency connected with the stability properties of the sheet. Fortunately, this effect hardly influences the overall quasi-steady character of the flow, at least in the Reynolds number domains observed so far.

5. The structure of the core of the spiral vortex sheet in incompressible flow

Experience indicates, and the theory presented by Howard & Matthews (1955) confirms, the important role of compressibility in the development of the core of the spiral. Nevertheless, a few remarks will be made on the spiral sheet in an incompressible fluid, as this question leads to a challenging mathematical problem of considerable theoretical interest. Previous treatments are due to Prandtl (1922), who gave the first formulation of the problem, together with some special solutions, and Kaden (1931), whose work represents the major basic contribution to our knowledge in this field.

The formulation of the problem as given in §4 is not yet complete. The condition (40) to be fulfilled along the spiral holds in this form only around $z_\lambda = 0$. For any point of the spiral, the following two conditions ('kinematic' and 'dynamic') have to be fulfilled:

(i) the normal velocity of the spiral sheet z_λ must equal the normal velocity of the fluid;

(ii) the pressure difference between the two sides of the sheet must vanish.

When the pressure along the spiral is computed, the time derivative of the potential (space-fixed) must be expressed by the time derivative along the spiral, i.e., along a point moving in accordance with the kinematic condition (i). The two conditions are easily formulated, but there exists a way of writing down both conditions directly in complex form, by use of Kirchhoff's law of the vortex motion, stating that every vortex element moves with a velocity equal to the regular part of the total velocity at the vortex point. The notions of this law have to be redefined for a continuous sheet.

It is well known that the regular part of the velocity is represented, in case of a sheet along a smooth curve, by the mean value of the velocities on both sides. The complex 'regular part' of the velocity will be

$$w_m = \tfrac{1}{2}(w_1 + w_2). \tag{44}$$

The meaning of a 'vortex point' for a continuous sheet z_λ has to be defined as a point of the sheet for which $\Gamma_\lambda = \text{constant}$, as an observer moving with such a point will never be 'passed' by a vortex element. (The definition $d\Gamma_\lambda/d\lambda=$ constant is obviously wrong, as the vortex elements can be stretched in the course of the sheet motion.) Thus, Kirchhoff's law, expressed by (21) for a single vortex, can be written for a continuous sheet in the following form:

$$\bar{w}_m=\left(\frac{dz_\lambda}{dt}\right)_{\Gamma_\lambda=\text{const.}}, \tag{45}$$

giving in a nutshell both kinematic and dynamic conditions.

Equation (45) will be applied to 'generalized' quasi-steady (i.e., similar) flows with time dependence expressed by (16). In this case,

$$\Gamma_\lambda=t^{2m-1}G\left(\frac{z_\lambda}{t^m}\right) \tag{46}$$

or

$$z_\lambda=t^m\zeta_\lambda\left(\frac{\Gamma_\lambda}{t^{2m-1}}\right), \tag{47}$$

so that (45) can be written as

$$w_m=t^{m-1}\left\{m\zeta_\lambda-(2m-1)G\frac{d\zeta_\lambda}{dG}\right\}. \tag{48}$$

The left-hand side has to be derived from the complex potential (42), putting $d\Gamma_\lambda=t^{2m-1}dG$, by finding the complex velocities on both sides of the spiral and taking the mean value. The computation of this quantity is naturally the difficult part of the problem, and (48) becomes a formidable non-linear singular integro-differential equation for ζ_λ as a function of G.

Kaden's results for the spiral core can be deduced and generalized by using a simple approximate expression for w_m in (48), namely,

$$w_m=-\frac{i\Gamma_\lambda}{2\pi z_\lambda}, \tag{49}$$

where z_λ (and ζ_λ) is now measured from the centre of the spiral. (It may be noted that (48) remains the same for any choice of 'similar' point as the origin.) Equation (49) expresses the assumption that w_m is the same as the velocity induced by a concentrated vortex of the strength Γ_λ (i.e., the circulation up to z_λ) situated at the spiral centre. It is easily seen that this assumption is reasonably good if the change of Γ_λ per turn of the spiral is small compared with Γ_λ.

Introduction of (49) into (48), by use of the variables defined in (46) and (47), leads to the equation

$$\frac{iG}{2\pi}=m\zeta_\lambda\bar{\zeta}_\lambda-(2m-1)G\bar{\zeta}_\lambda\frac{d\zeta_\lambda}{dG}. \tag{50}$$

For the solution, put

$$\zeta_\lambda=s(G)\exp i\theta(G).$$

Introduction of this expression into (50) leads, upon splitting into real and imaginary parts, to the following two equations:

$$ms^2-(2m-1)Gss'=0,$$

$$-(2m-1)s^2\theta'=\frac{G}{2\pi}.$$

After an easy integration, the final result is

$$\zeta_\lambda = BG^{m/(2m-1)} \exp\left\{\frac{iG^{-1/(2m-1)}}{2\pi B^2}\right\}, \tag{51}$$

where B is a constant of integration. The connection between the absolute value and the argument of ζ_λ is the following:

$$s = B^{1-2m}(2\pi\theta)^{-m}. \tag{52}$$

For $m=1$, the equation of the spiral in incompressible flow would be, in Kaden's approximation, $s\theta = \text{constant}$, for $\theta \to \infty$.

Expressed by the radius s, it is found that

$$G = \left(\frac{s}{B}\right)^{(2m-1)/m}, \tag{53}$$

$$\frac{|W_m|}{t^{m-1}} = \frac{1}{2\pi B}\left(\frac{s}{B}\right)^{(m-1)/m}, \tag{54}$$

and on introducing a spiral arc length $d\lambda$ for the almost circular core by the approximate equation

$$d\lambda = s\,d\theta, \tag{55}$$

it can be shown with the help of (52) that

$$\frac{dG}{d\lambda} = 2\pi(2m-1)s. \tag{56}$$

The change of G per turn becomes proportional to s^2, which for small values of s will always be less than $G \sim s^{(2m-1)/m}$.

The singular behaviour in the case of $m=\frac{1}{2}$ has been pointed out before: the 'concentrated vortex' solution is exact, and there is no spiral. The value $m=\frac{1}{2}$ of the exponent must represent a limiting case for possible 'similar' flows of this type, as, for $m<\frac{1}{2}$, G becomes infinite at the centre $s=0$ and decreases with s, i.e., vorticity flows back from the core to the generating edge!

It has been noted before that for $\beta=0$ the leading portion of the spiral can be handled; now approximations for the central core in incompressible flow have been given. The remaining task of 'joining' the two solutions, and ultimately of finding the potential for the whole flow field, will certainly not be easy.

References

Brown, C. E. & Michael, W. H. 1954 Effect of leading edge separation on the lift of a delta wing, *J. Aero. Sci.* **21**, 690.

Cheng, H. K. 1955 Aerodynamics of a rectangular plate with vortex separation in supersonic flow, *J. Aero. Sci.* **22**, 217.

Edwards, R. H. 1954 Leading edge separation from delta wings, *J. Aero. Sci.* **21**, 134.

FRIEDLANDER, F. G. 1946 The diffraction of sound pulses, *Proc. Roy. Soc.* A **186,** 322.

HOWARD, L. N. & MATTHEWS, D. L. 1955 On the vortices produced in shock diffraction, Technical Report 11–21, Princeton University.

KADEN, H. 1931 Aufwickling einer unstabilen Unstetigkeitsfläche, *Ing. Arch.* **2,** 140.

KELLER, J. B. & BLENK, A. 1951 Diffraction and reflection of pulses by wedges and corners, *Comm. Pure Appl. Math.* **4,** 75.

MICHAEL, W. H. 1955 Flow studies on flat-plate delta wings at supersonic speed, *Nat. Adv. Comm. Aero., Wash., Tech. Note* no. 3472.

MILES, J. W. 1952 On the diffraction of an acoustic pulse by a wedge, *Proc. Roy. Soc.* A, **212,** 543.

PRANDTL, L. 1922 Über die Entstehung von Wirbeln in der idealen Flüssigkeit, *Vorträge aus dem Gebiete der Hydro- und Aerodynamik* (Editors, Th. v. Karman and T. Levi-Civita), Innsbruck.

UHLENBECK, G. 1950 Diffraction of shock waves around various obstacles, Report on Project M720–4, Engineering Research Institute, University of Michigan.

WALDRON, H. F. 1954 An experimental study of a spiral vortex formed by shock-wave diffraction, Technical Note no. 2, Institute of Aerophysics, University of Toronto.